

# ON MOMENTS OF $|\zeta(\frac{1}{2} + it)|$ IN SHORT INTERVALS

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*Dedicated to Prof. K. Ramachandra on the occasion of his seventieth birthday*

ABSTRACT. Power moments of

$$J_k(t, G) = \frac{1}{\sqrt{\pi G}} \int_{-\infty}^{\infty} |\zeta(\frac{1}{2} + it + iu)|^{2k} e^{-(u/G)^2} du \quad (t \asymp T, T^\varepsilon \leq G \ll T),$$

where  $k$  is a natural number, are investigated. The results that are obtained are used to show how bounds for  $\int_0^T |\zeta(\frac{1}{2} + it)|^{2k} dt$  may be obtained.

## 1. INTRODUCTION

Power moments represent one of the most important parts of the theory of the Riemann zeta-function  $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$  ( $\sigma = \Re s > 1$ ). Of particular significance are the moments on the “critical line”  $\sigma = \frac{1}{2}$ , and a vast literature exists on this subject (see e.g., [8], [9], [20], [22], [24] and [26]). Let us define

$$(1.1) \quad I_k(T) = \int_0^T |\zeta(\frac{1}{2} + it)|^{2k} dt,$$

where  $k \in \mathbb{R}$  is a fixed, positive number. Naturally one would want to find an asymptotic formula for  $I_k(T)$  for a given  $k$ , but this is an extremely difficult problem. Except when  $k = 1$  and  $k = 2$ , no asymptotic formula for  $I_k(T)$  is known yet, although there are plausible conjectures for such formulas (see e.g., [2]). In the absence of asymptotic formulas for  $I_k(T)$ , one would like then to obtain upper and lower bounds for  $I_k(T)$ , and for the closely related problem of

$$(1.2) \quad I_k(T + G) - I_k(T - G) = \int_{T-G}^{T+G} |\zeta(\frac{1}{2} + it)|^{2k} dt \quad (1 \ll G \leq T).$$

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For the latter, important results were obtained by K. Ramachandra, either alone, or in collaboration with R. Balasubramanian. Many of his results are contained in his comprehensive monograph [24] on mean values and omega-results for the Riemann zeta-function. In particular, [24] contains the proof of the lower bound

$$(1.3) \quad \int_{T-G}^{T+G} |\zeta(\tfrac{1}{2} + it)|^{2k} dt \gg_k G(\log G)^{k^2} \quad (\log \log T \ll_k G \leq T, k \in \mathbb{N}),$$

where  $\ll_k$  (or  $\gg_k$ ) means that the implied constant depends only on  $k$ . One believes that the bound in (1.3) represents the correct order of magnitude, at least for a certain range of  $G$  for a given  $k \in \mathbb{N}$ . Unfortunately, even proving the corresponding much weaker upper bound (for  $G = T$ ), namely

$$(1.4) \quad I_k(T) \ll_{\varepsilon, k} T^{1+\varepsilon} \quad (k > 0)$$

seems at present impossible for any  $k > 2$ . Here and later,  $\varepsilon > 0$  denotes constants which may be arbitrarily small, but are not necessarily the same ones at each occurrence. In view of the relation (see [9] or [24])

$$(1.5) \quad \zeta^k(\tfrac{1}{2} + it) \ll_k \log t \left( \int_{t-1/3}^{t+1/3} |\zeta(\tfrac{1}{2} + iu)|^k dt \right) + 1 \quad (k \in \mathbb{N}),$$

it is easily seen that (1.4) (for all  $k$ ) is equivalent to the famous *Lindelöf hypothesis* that  $\zeta(\tfrac{1}{2} + it) \ll_\varepsilon |t|^\varepsilon$ . The Lindelöf hypothesis, like the even more famous *Riemann hypothesis* (that all complex zeros of  $\zeta(s)$  have real part  $1/2$ ), is neither proved nor disproved at the time of writing of this text. For a discussion on this subject, see [13].

The aim of this paper is to investigate upper bounds for  $I_k(T)$  when  $k \in \mathbb{N}$ , which we henceforth assume. The problem can be reduced to bounds of  $|\zeta(\tfrac{1}{2} + it)|$  over short intervals, as in (1.2), but it is more expedient to work with the smoothed integral

$$(1.6) \quad J_k(T, G) := \frac{1}{\sqrt{\pi G}} \int_{-\infty}^{\infty} |\zeta(\tfrac{1}{2} + iT + iu)|^{2k} e^{-(u/G)^2} du \quad (1 \ll G \ll T).$$

Namely we obviously have

$$(1.7) \quad I_k(T+G) - I_k(T-G) = \int_{-G}^G |\zeta(\tfrac{1}{2} + iT + iu)|^{2k} du \leq \sqrt{\pi} e G J_k(T, G),$$

and it is technically more convenient to work with  $J_k(T, G)$  than with  $I_k(T+G) - I_k(T-G)$ . Of course, instead of the Gaussian exponential weight  $\exp(-(u/G)^2)$ ,

one could introduce in (1.6) other smooth weights with a similar effect. The Gaussian weight has the advantage that, by the use of the classical integral

$$(1.8) \quad \int_{-\infty}^{\infty} \exp(Ax - Bx^2) dx = \sqrt{\frac{\pi}{B}} \exp\left(\frac{A^2}{4B}\right) \quad (\Re B > 0),$$

one can often explicitly evaluate the relevant exponential integrals that appear in the course of the proof.

The plan of the paper is as follows. In the next two sections we shall briefly discuss the results on  $I_k(T)$  and  $J_k(T, G)$  when  $k = 1$  and  $k = 2$ , respectively. Indeed, as these are the only cases when we possess relatively good knowledge and explicit formulas, it is only natural that those results be used in deriving results on higher power moments, when our knowledge is quite imperfect. We shall obtain new results on moments of  $J_k(T, G)$  by using the explicit formulas of Section 2 and Section 3. This will be done in Section 4 and Section 5. Finally, in Section 6, it will be shown how one can obtain bounds for  $I_k(T)$  from the bounds of moments of  $J_k(T, G)$ .

## 2. THE MEAN SQUARE FORMULA

The mean square formula for  $|\zeta(\frac{1}{2} + it)|$  is traditionally written in the form

$$(2.1) \quad \int_0^T |\zeta(\frac{1}{2} + it)|^2 dt = T \log\left(\frac{T}{2\pi}\right) + (2\gamma - 1)T + E(T),$$

where  $\gamma = -\Gamma'(1) = 0.577 \dots$  is Euler's constant, and  $E(T)$  is to be considered as the error term in the asymptotic formula (2.1). F.V. Atkinson [1] established in 1949 an explicit, albeit complicated formula for  $E(T)$ , containing two exponential sums of length  $\asymp T$  weighted by the number of divisors function  $d(n)$ , plus an error term which is  $O(\log^2 T)$ . This is given as

LEMMA 1. *Let  $0 < A < A'$  be any two fixed constants such that  $AT < N < A'T$ , and let  $N' = N'(T) = T/(2\pi) + N/2 - (N^2/4 + NT/(2\pi))^{1/2}$ . Then*

$$(2.2) \quad E(T) = \Sigma_1(T) + \Sigma_2(T) + O(\log^2 T),$$

where

$$(2.3) \quad \Sigma_1(T) = 2^{1/2}(T/(2\pi))^{1/4} \sum_{n \leq N} (-1)^n d(n) n^{-3/4} e(T, n) \cos(f(T, n)),$$

$$(2.4) \quad \Sigma_2(T) = -2 \sum_{n \leq N'} d(n) n^{-1/2} (\log T/(2\pi n))^{-1} \cos(T \log T/(2\pi n) - T + \pi/4),$$

with

(2.5)

$$\begin{aligned} f(T, n) &= 2T \operatorname{arsinh} \left( \sqrt{\pi n / (2T)} \right) + \sqrt{2\pi n T + \pi^2 n^2} - \frac{1}{4}\pi \\ &= -\frac{1}{4}\pi + 2\sqrt{2\pi n T} + \frac{1}{6}\sqrt{2\pi^3} n^{3/2} T^{-1/2} + a_5 n^{5/2} T^{-3/2} + a_7 n^{7/2} T^{-5/2} + \dots, \end{aligned}$$

$$\begin{aligned} (2.6) \quad e(T, n) &= (1 + \pi n / (2T))^{-1/4} \left\{ (2T / \pi n)^{1/2} \operatorname{arsinh} \left( \sqrt{\pi n / (2T)} \right) \right\}^{-1} \\ &= 1 + O(n/T) \quad (1 \leq n < T), \end{aligned}$$

and  $\operatorname{arsinh} x = \log(x + \sqrt{1 + x^2})$ .

Atkinson's formula was the starting point for many results on  $E(T)$  (see [8, Chapter 15] for some of them). It is conjectured that  $E(T) \ll_{\varepsilon} T^{1/4+\varepsilon}$ , but currently this bound cannot be proved even if the Riemann Hypothesis is assumed. The best known upper bound for  $E(T)$ , obtained by intricate estimation of a certain exponential sum, is due to M.N. Huxley [6]. This is

$$E(T) \ll T^{72/227} (\log T)^{679/227}, \quad \frac{72}{227} = 0.3171806 \dots$$

In the other direction, J.L. Hafner and the author [3] proved that there exist absolute constants  $A, B > 0$  such that

$$E(T) = \Omega_+ \left\{ (T \log T)^{1/4} (\log \log T)^{(3+\log 4)/4} e^{-A\sqrt{\log \log \log T}} \right\}$$

and

$$E(T) = \Omega_- \left\{ T^{1/4} \exp \left( \frac{B(\log \log T)^{1/4}}{(\log \log \log T)^{3/4}} \right) \right\},$$

where  $f(x) = \Omega_+(g(x))$  means that  $\limsup_{x \rightarrow \infty} f(x)/g(x) > 0$ , and  $f(x) = \Omega_-(g(x))$  means that  $\liminf_{x \rightarrow \infty} f(x)/g(x) < 0$ .

In what follows we shall formulate an explicit formula for  $J_1(T, G)$ . Such a result can be, of course, deduced from Atkinson's formula (2.3)-(2.5) by the use of (1.8). This approach was used originally by D.R. Heath-Brown [4], who proved

$$(2.7) \quad \int_0^T |\zeta(\tfrac{1}{2} + it)|^{12} dt \ll T^2 \log^{17} T,$$

which is still essentially the best known result concerning higher power moments of  $|\zeta(\frac{1}{2} + it)|$ . This procedure can be avoided by appealing to Y. Motohashi's formula [22, p. 213], which states that

(2.8)

$$J_1(T, G) = 2^{\frac{3}{4}} \pi^{\frac{1}{4}} T^{-\frac{1}{4}} \sum_{n=1}^{\infty} (-1)^n d(n) n^{-\frac{1}{4}} \sin f(T, n) \exp\left(-\frac{\pi n G^2}{2T}\right) + O(\log T),$$

where  $f(T, n)$  is given by (2.5), and  $T^{1/4} \leq G \leq T/\log T$ . In fact, only the range  $G \leq T^{1/3}$  is relevant, since for  $G \geq T^{1/3}$  one has  $J_1(T, G) \ll \log T$  by [8, Chapter 7]. Motohashi's proof of (2.8), like the proof of Atkinson's formula for  $E(T)$ , is based on classical methods from analytic number theory. Albeit the expression on the right-hand side of (2.8) is quite simple, the condition  $G \geq T^{1/4}$  is rather restrictive for the application that we have in mind. Thus we shall use a similar type of result, which is valid in a much wider range. This is contained in

LEMMA 2. *For  $T^\varepsilon \leq G \leq T$  and  $f(T, n)$  given by (2.5), we have*

$$(2.9) \quad \begin{aligned} J_1(T, G) &= O(\log T) + \\ &+ \sqrt{2} \sum_{n=1}^{\infty} (-1)^n d(n) n^{-1/2} \left( \left( \frac{T}{2\pi n} + \frac{1}{4} \right)^{1/2} - \frac{1}{2} \right)^{-1/2} \times \\ &\times \exp\left(-G^2 (\operatorname{arsinh} \sqrt{\pi n/(2T)})^2\right) \sin f(T, n). \end{aligned}$$

By using Taylor's formula it is seen that the error made by replacing

$$\left( \left( \frac{T}{2\pi n} + \frac{1}{4} \right)^{1/2} - \frac{1}{2} \right)^{-1/2} \exp\left(-G^2 (\operatorname{arsinh} \sqrt{\pi n/(2T)})^2\right)$$

by

$$\left( \frac{T}{2\pi n} \right)^{-1/4} \exp(-\pi G^2/(nT))$$

is  $\ll 1$  for  $G \geq T^{1/5} \log^C T$ . But the important fact is that in applications (2.9) is as useful as (2.8), since the factors under the sine function are identical.

**Proof of Lemma 2.** The proof of (2.9) follows from Y. Motohashi [22, Theorem 4.1], which gives that

(2.10)

$$\begin{aligned} \int_{-\infty}^{\infty} |\zeta(\tfrac{1}{2} + it)|^2 g(t) dt &= \int_{-\infty}^{\infty} \left[ \Re \left\{ \frac{\Gamma'}{\Gamma}(\tfrac{1}{2} + it) \right\} + 2\gamma - \log(2\pi) \right] g(t) dt \\ &+ 2\pi \Re(g(\tfrac{1}{2}i)) + 4 \sum_{n=1}^{\infty} d(n) \int_0^{\infty} (y(y+1))^{-1/2} g_c(\log(1+1/y)) \cos(2\pi ny) dy, \end{aligned}$$

where

$$g_c(x) := \int_{-\infty}^{\infty} g(t) \cos(xt) dt$$

is the cosine Fourier transform of  $g(t)$ . One requires the function  $g(r)$  to be real-valued for  $r \in \mathbb{R}$ , and that there exists a large constant  $A > 0$  such that  $g(r)$  is regular and  $\ll (|r| + 1)^{-A}$  for  $|\Im r| \leq A$ . The choice

$$g(t) = \frac{1}{\sqrt{\pi}G} e^{-(T-t)^2/G^2}, \quad g_c(x) = e^{-\frac{1}{4}(Gx)^2} \cos(Tx)$$

is permissible, and then the integral on the left-hand side of (2.10) becomes  $J_1(T, G)$ . The first integral on the right-hand side of (2.10) is  $O(\log T)$ , and the second one is evaluated by the saddle-point method (see e.g., [8, Chapter 2]). A convenient result to use is [8, Theorem 2.2 and Lemma 15.1], due originally to Atkinson [1]. In the latter only the exponential factor  $\exp(-\frac{1}{4}G^2 \log(1 + 1/y))$  is missing. In the notation of [1] and [8] we have that the saddle point  $x_0$  satisfies

$$x_0 = U - \frac{1}{2} = \left( \frac{T}{2\pi n} + \frac{1}{4} \right)^{1/2} - \frac{1}{2},$$

and the presence of the above exponential factor makes it possible to truncate the series in (2.9) at  $n = TG^{-2} \log T$  with a negligible error. Furthermore, in the remaining range for  $n$  we have

$$\Phi_0 \mu_0 F_0^{-3/2} \ll (nT)^{-3/4},$$

which makes a total contribution of  $O(1)$ , as does error term integral in Theorem 2.2 of [8]. The error terms with  $\Phi(a)$ ,  $\Phi(b)$  vanish for  $a = 0$ ,  $b = \infty$ , and (2.9) follows.

### 3. THE FORMULA FOR THE FOURTH MOMENT

The asymptotic formula for the fourth moment of the Riemann zeta-function  $\zeta(s)$  on the critical line is customarily written as

$$(3.1) \quad \int_0^T |\zeta(\tfrac{1}{2} + it)|^4 dt = TP_4(\log T) + E_2(T), \quad P_4(x) = \sum_{j=0}^4 a_j x^j.$$

A classical result of A.E. Ingham [7] from 1926 is that  $a_4 = 1/(2\pi^2)$  and that the error term  $E_2(T)$  in (3.1) satisfies the bound  $E_2(T) \ll T \log^3 T$  (a simple proof

of this is due to K. Ramachandra [23]). Much later D.R. Heath-Brown [4] made progress in this problem by proving that  $E_2(T) \ll_\varepsilon T^{7/8+\varepsilon}$ . He also calculated

$$a_3 = 2(4\gamma - 1 - \log(2\pi) - 12\zeta'(2)\pi^{-2})\pi^{-2}$$

and produced more complicated expressions for  $a_0, a_1$  and  $a_2$  in (3.1). For an explicit evaluation of the  $a_j$ 's in (3.1) the reader is referred to the author's work [11]. In the last fifteen years, due primarily to the application of powerful methods of spectral theory (see Y. Motohashi's monograph [22] for a comprehensive account), much advance has been made in connection with  $E_2(T)$ . This involves primarily results with exponential sums involving the quantities  $\kappa_j$  and  $\alpha_j H_j^3(\frac{1}{2})$ . Here as usual  $\{\lambda_j = \kappa_j^2 + \frac{1}{4}\} \cup \{0\}$  is the discrete spectrum of the non-Euclidean Laplacian acting on  $SL(2, \mathbb{Z})$ -automorphic forms, and  $\alpha_j = |\rho_j(1)|^2 (\cosh \pi \kappa_j)^{-1}$ , where  $\rho_j(1)$  is the first Fourier coefficient of the Maass wave form corresponding to the eigenvalue  $\lambda_j$  to which the Hecke series  $H_j(s)$  is attached. It is conjectured that  $E_2(T) \ll_\varepsilon T^{1/2+\varepsilon}$ , which would imply the (hitherto unproved) bound  $\zeta(\frac{1}{2} + it) \ll_\varepsilon |t|^{1/8+\varepsilon}$ . It is known now that

$$(3.2) \quad E_2(T) = O(T^{2/3} \log^{C_1} T), \quad E_2(T) = \Omega(T^{1/2}),$$

$$(3.3) \quad \int_0^T E_2(t) dt = O(T^{3/2}), \quad \int_0^T E_2^2(t) dt = O(T^2 \log^{C_2} T),$$

with effective constants  $C_1, C_2 > 0$  (the values  $C_1 = 8, C_2 = 22$  are worked out in [22]). The above results were proved by Y. Motohashi and the author: (3.2) and the first bound in (3.3) in [17], and the second upper bound in (3.3) in [16]. The  $\Omega$ -result in (3.2) was improved to  $E_2(T) = \Omega_\pm(T^{1/2})$  by Y. Motohashi [21]. It turns out that there is no explicit formula for  $E_2(T)$  which would represent the analogue of Atkinson's formula (cf. Lemma 1). Results on  $E_2(T)$  have been obtained indirectly, by using the explicit formula for  $J_2(T, G)$ , due to Y. Motohashi (see [22]). This is

LEMMA 3. *Let  $D > 0$  be an arbitrary constant. For  $T^{1/2} \log^{-D} T \leq G \leq T/\log T$  we have*

$$(3.4) \quad \begin{aligned} J_2(T, G) &= O(\log^{3D+9} T) \\ &+ \frac{\pi}{\sqrt{2T}} \sum_{j=1}^{\infty} \alpha_j H_j^3(\tfrac{1}{2}) \kappa_j^{-1/2} \sin\left(\kappa_j \log \frac{\kappa_j}{4eT}\right) \exp(-\tfrac{1}{4}(G\kappa_j/T)^2). \end{aligned}$$

In what concerns higher moments, let us only state that from (2.7) and (3.1) one obtains by Hölders's inequality for integrals

$$(3.5) \quad \int_0^T |\zeta(\tfrac{1}{2} + it)|^6 dt \ll T^{5/4} \log^{29/4} T, \quad \int_0^T |\zeta(\tfrac{1}{2} + it)|^8 dt \ll T^{3/2} \log^{21/2} T.$$

The bounds in (3.5) are hitherto the sharpest ones known.

Let it be also mentioned here that the sixth moment was investigated by the author in [12], where it was shown that

$$(3.6) \quad \int_0^T |\zeta(\tfrac{1}{2} + it)|^6 dt \ll_{\varepsilon} T^{1+\varepsilon}$$

does hold if a certain conjecture involving the so-called ternary additive divisor problem is true.

#### 4. THE MOMENTS OF $J_1(t, G)$

In this section we shall prove results on moments of  $J_1(t, G)$ . One expects this function, at least for certain ranges of  $G$ , to behave like  $O(t^{\varepsilon})$  on the average. Our bounds are contained in

**THEOREM 1.** *We have*

$$(4.1) \quad \int_T^{2T} J_1^m(t, G) dt \ll_{\varepsilon} T^{1+\varepsilon}$$

for  $T^{\varepsilon} \leq G \leq T$  if  $m = 1, 2$ ; for  $T^{1/7+\varepsilon} \leq G \leq T$  if  $m = 3$ , and for  $T^{1/5+\varepsilon} \leq G \leq T$  if  $m = 4$ .

**Proof of Theorem 1.** Our starting point in all cases is the explicit formula (2.9). The results for  $m = 1$  and  $m = 2$  follow by straightforward integration and the first derivative test for exponential integrals (see [8, Lemma 2.1]). The proof resembles mean square bounds for  $\Delta(x)$  (the error term in the divisor problem) and  $E(t)$  (op. cit.), and is omitted for the sake of brevity. Instead, we shall concentrate on the more difficult cases  $m = 3$  and  $m = 4$ . For this we shall need two lemmas on the spacing of three and four square roots (the square roots appear in view of the asymptotic formula given in (2.5)). These are

**LEMMA 4.** *Let  $\mathcal{N}$  denote the number of solutions in integers  $m, n, k$  of the inequality*

$$|\sqrt{m} + \sqrt{n} - \sqrt{k}| \leq \delta \sqrt{M} \quad (\delta > 0)$$



with  $M' < n \leq 2M'$ ,  $M < m \leq 2M$ , and  $M' \leq M$ . Then

$$(4.2) \quad \mathcal{N} \ll_{\varepsilon} M^{\varepsilon} (M^2 M' \delta + (MM')^{1/2}).$$

LEMMA 5 *Let  $k \geq 2$  be a fixed integer and  $\delta > 0$  be given. Then the number of integers  $n_1, n_2, n_3, n_4$  such that  $N < n_1, n_2, n_3, n_4 \leq 2N$  and*

$$|n_1^{1/k} + n_2^{1/k} - n_3^{1/k} - n_4^{1/k}| < \delta N^{1/k}$$

*is, for any given  $\varepsilon > 0$ ,*

$$(4.3) \quad \ll_{\varepsilon} N^{\varepsilon} (N^4 \delta + N^2).$$

Lemma 4 was proved by Sargos and the author [18], while Lemma 5 is due to Robert–Sargos [25]. The plan of the proof of (4.1) when  $m = 3$  is simple: the expression (2.9) will be raised to the third power and then integrated. There are, however, two obstacles in attaining this goal. The first is that direct integration does not lead to adequate truncation, so that some smoothing of the relevant integral will be made. The second one is that in the asymptotic formula for  $f(T, n)$  in (2.5) not only square roots appear, but also higher powers. To get around this difficulty we shall appeal to

LEMMA 6 (M. Jutila [9]). *For  $A \in \mathbb{R}$  we have*

$$(4.4) \quad \cos\left(\sqrt{8\pi n T} + \frac{1}{6}\sqrt{2\pi^3 n}^{3/2} T^{-1/2} - A\right) = \int_{-\infty}^{\infty} \alpha(u) \cos(\sqrt{8\pi n}(\sqrt{T} + u) - A) du,$$

*where  $\alpha(u) \ll T^{1/6}$  for  $u \neq 0$ ,*

$$(4.5) \quad \alpha(u) \ll T^{1/6} \exp(-bT^{1/4}|u|^{3/2})$$

*for  $u < 0$ , and*

$$(4.6) \quad \alpha(u) = T^{1/8} u^{-1/4} \left( d \exp(ibT^{1/4} u^{3/2}) + \bar{d} \exp(-ibT^{1/4} u^{3/2}) \right) + O(T^{-1/8} u^{-7/4})$$

*for  $u \geq T^{-1/6}$  and some constants  $b (> 0)$  and  $d$ .*

Now we continue with the proof of Theorem 1. Write first

$$(4.7) \quad \int_T^{2T} J_1^m(t, G) dt \leq \int_{T/2}^{5T/2} \varphi(t) J_1^m(t, G) dt,$$

where  $\varphi(t)$  ( $\geq 0$ ) is a smooth function supported in  $[T/2, 5T/2]$  such that  $\varphi(t) = 1$  when  $t \in [T, 2T]$ , and then we have  $\varphi^{(r)}(t) \ll_r T^{-r}$  ( $r = 0, 1, 2, \dots$ ). We truncate (2.9) at  $TG^{-2} \log T$  and use it to expand the  $m$ -th power on the right-hand side of (4.7) when  $m = 3, 4$ . The terms  $a_{2j-2} n^{(2j-1)/2} T^{-j/2}$  ( $j \geq 3$ ) in  $f(T, n)$  (cf. (2.5)) are expanded by Taylor's formula. Since

$$n^{5/2} t^{-3/2} \ll TG^{-5} \log^{5/2} T \leq T^{-\varepsilon}$$

for  $G \geq T^{1/5+\varepsilon}$ , it transpires that in this range for  $G$  we may take sufficiently many terms in Taylor's formula so that the error term will make a negligible contribution. The other terms will lead to similar expressions, and the largest contribution will come from the constant term. After this there will remain

$$\sin h(t, k) \sin h(t, \ell) \sin h(t, n), \quad h(t, u) = \sqrt{8\pi t u} + \frac{1}{6} \sqrt{2\pi^3} u^{3/2} t^{-1/2} - \frac{1}{4} \pi$$

when  $m = 3$ , with  $T^{1/3} < k, \ell, n \leq TG^{-2} \log T$ ,  $k \asymp K$ ,  $k \geq \max(\ell, n)$ . The factors with  $u^{3/2}$  (for  $u = k, \ell, n$ ) are removed from the  $h$ -functions by the use of Lemma 6. With  $\alpha(v)$  given by (4.6) we have

$$\begin{aligned} \cos \left( \sqrt{8\pi n t} + \frac{1}{6} \sqrt{2\pi^3} n^{3/2} t^{-1/2} - A \right) &= O(T^{-10}) + \\ &\int_{-u_0}^{u_1} \alpha(v) \cos(\sqrt{8\pi n}(\sqrt{t} + v) - A) dv + \int_{u_1}^{\infty} \alpha(v) \cos(\sqrt{8\pi n}(\sqrt{t} + v) - A) dv, \end{aligned}$$

where we set

$$(4.9) \quad u_0 = T^{-1/6} \log T, \quad u_1 = CKT^{-1/2},$$

and  $C > 0$  is a large constant. With this choice of  $u_0, u_1$  and (4.5)-(4.6) it follows that, for  $T/2 \leq t \leq 5T/2$ ,

$$(4.10) \quad \int_{-u_0}^{u_1} \alpha(v) \cos(\sqrt{8\pi n}(\sqrt{t} + v) - A) dv + \int_{u_1}^{\infty} \alpha(v) \cos(\sqrt{8\pi n}(\sqrt{t} + v) - A) dv \ll \log T.$$

Namely we have

$$\int_{u_0}^{u_1} t^{1/8} v^{-1/4} \exp(ibt^{1/4} v^{3/2} \pm \sqrt{8\pi n} v) dv \ll \log T,$$

on writing the integral as a sum of  $\ll \log T$  integrals over  $[U, U']$  with  $u_0 \leq U < U' \leq 2U \ll u_1$ , and applying the second derivative test (i.e., [8, Lemma 2.2]) to

each of these integrals. We remark that the contribution of the  $O$ -term in (4.6) will be, by trivial estimation,  $O(1)$ . It remains yet to deal with the integral with  $v > u_1$  in (4.10), when we note that

$$\frac{\partial}{\partial v} \left( bt^{1/4}v^{3/2} \pm \sqrt{8\pi nv} \right) \gg T^{1/4}v^{1/2} \quad (v > u_1, t \asymp T),$$

provided that  $C$  in (4.9) is sufficiently large. Thus by the first derivative test

$$\begin{aligned} & \int_{u_1}^{\infty} \alpha(v) \cos(\sqrt{8\pi n}(\sqrt{t} + v) - A) dv \\ & \ll 1 + T^{1/8}u_1^{-1/4}T^{-1/4}u_1^{-1/2} \\ & \ll 1 + T^{1/4}K^{-3/4} \ll 1, \end{aligned}$$

since  $K \gg T^{1/3}$ . Thus (4.10) holds.

Hence setting

$$E_{\pm} := \sqrt{8\pi}(\sqrt{k} + \sqrt{\ell} \pm \sqrt{n}),$$

it is seen that we are left with the integral of

$$(4.11) \quad \int_{T/2}^{5T/2} \varphi(t) F(t; k, \ell, n) e^{iE_{\pm}\sqrt{t}} dt,$$

where

$$\begin{aligned} F(t; k, \ell, n) &:= \left( \left( \frac{t}{2\pi k} + \frac{1}{4} \right)^{1/2} - \frac{1}{2} \right)^{-1/2} \left( \left( \frac{t}{2\pi \ell} + \frac{1}{4} \right)^{1/2} - \frac{1}{2} \right)^{-1/2} \\ &\times \left( \left( \frac{t}{2\pi n} + \frac{1}{4} \right)^{1/2} - \frac{1}{2} \right)^{-1/2} \exp(-G^2(\operatorname{arsinh} \sqrt{\pi k/(2t)})^2) \\ &\times \exp(-G^2(\operatorname{arsinh} \sqrt{\pi \ell/(2t)})^2) \exp(-G^2(\operatorname{arsinh} \sqrt{\pi n/(2t)})^2). \end{aligned}$$

Repeated integration by parts show that the integral in (4.11) with  $E_+$  will make a negligible contribution, and also the one with  $|E_-| \geq T^{\varepsilon-1/2}$  for any given  $\varepsilon > 0$ . The contribution of those  $k, \ell, n$  for which  $|E_-| \leq T^{\varepsilon-1/2}$  is estimated by the use of Lemma 4 (with an obvious change of notation and with  $\delta \asymp K^{-1/2}T^{\varepsilon-1/2}$ ). After this, the integral over  $t$  is estimated trivially, and (4.10) is used. The relevant expression on the right-hand side of (4.7) is

$$\begin{aligned} & \ll_{\varepsilon} T^{1+\varepsilon} \max_{T^{1/3} \leq K \leq TG^{-2} \log T} (TK)^{-3/4} (K^3 \cdot K^{-1/2}T^{-1/2} + K) \\ & \ll_{\varepsilon} T^{\varepsilon} \max_{K \leq TG^{-2} \log T} (K^{7/4}T^{-1/4} + T^{1/4}K^{1/4}) \\ & \ll_{\varepsilon} T^{3/2+\varepsilon} G^{-7/2} + T^{1/2+\varepsilon} \ll_{\varepsilon} T^{1+\varepsilon} \end{aligned}$$

for  $G \geq T^{1/7+\varepsilon}$ . However, our initial condition was  $G \geq T^{1/5+\varepsilon}$ , which is more restrictive. Fortunately, this is a technical point that can be resolved by modifying Lemma 6 suitably. Namely, instead of  $n^{3/2}T^{-1/2}$  in (4.4) we may put  $Cn^{5/2}T^{-3/2}$ , which will be “removed” in the fashion of Lemma 6. Instead of the function  $\alpha(u)$ , another oscillating function  $\beta(u)$  will appear, for which the analogue of (4.8) will hold. Jutila obtained (4.5)-(4.6) by exploiting the fact that the inversion made in (4.4) can be connected to the *Airy integral*

$$\text{Ai}(x) := \frac{1}{\pi} \int_0^\infty \cos(\tfrac{1}{3}t^3 + tx) dt \quad (x \geq 0),$$

for which there exist representations in terms of the classical Bessel functions, thereby providing quickly asymptotic expansions necessary in Lemma 6. In the new case there will be no Airy integrals involved, but the necessary asymptotic expansion can be obtained by the use of the saddle point method.

This ends the discussion of the case  $m = 3$ . The case  $m = 4$  will be analogous, the non-trivial contribution will come from integer quadruples  $(n_1, n_2, n_3, n_4)$  such that  $K < n_1, n_2, n_3, n_4 \leq 2K$  ( $T^{1/3} \leq K \leq TG^{-2} \log T$ ) and

$$(4.12) \quad |\sqrt{n_1} + \sqrt{n_2} - \sqrt{n_3} - \sqrt{n_4}| \leq T^{\varepsilon-1/2}.$$

Instead of Lemma 4 we use Lemma 5 (with  $k = 2$ ), to obtain a contribution which is

$$\begin{aligned} &\ll_\varepsilon T^{1+\varepsilon} \max_{K \leq TG^{-2} \log T} (TK)^{-1} (K^4 \cdot K^{-1/2} T^{-1/2} + K^2) \\ &\ll_\varepsilon T^\varepsilon \max_{K \leq TG^{-2} \log T} (T^{-1/2} K^{5/2} + K) \\ &\ll_\varepsilon T^{2+\varepsilon} G^{-5} + T^{1+\varepsilon} \ll_\varepsilon T^{1+\varepsilon} \end{aligned}$$

for  $G \geq T^{1/5+\varepsilon}$ , as asserted. In this case direct application of Lemma 6 suffices. Values of  $m$  satisfying  $m > 4$  in (4.1) could be handled in a similar fashion, provided that one can find analogues of Lemma 4 and Lemma 5 which are strong enough.

## 5. THE MOMENTS OF $J_2(t, G)$

We shall prove now the analogue of Theorem 1 for  $J_2(t, G)$ . This is a more difficult problem, and the ranges for  $G$  for which the analogue of (4.1) will hold will be poorer. The result is

**THEOREM 2.** *We have*

$$(5.1) \quad \int_T^{2T} J_2^m(t, G) dt \ll_\varepsilon T^{1+\varepsilon}$$

for  $T^{1/2+\varepsilon} \leq G \leq T$  if  $m = 1, 2$ ; for  $T^{4/7+\varepsilon} \leq G \leq T$  if  $m = 3$ , and for  $T^{3/5+\varepsilon} \leq G \leq T$  if  $m = 4$ .

**Proof of Theorem 1.** Our starting point is Lemma 3. We remark that by using the estimate (see Y. Motohashi [22, Section 3.4])

$$(5.2) \quad \sum_{\kappa_j \leq K} \alpha_j H_j^3(\tfrac{1}{2}) \ll_{\varepsilon} K^2 \log^3 K$$

we see that for  $G \geq T^{2/3} \log^C T$  the right-hand side of (3.4) is  $O(1)$ , hence we may suppose that  $T^{1/2+\varepsilon} \leq G \leq T^{2/3} \log^C T$ . Observe also that we may truncate the series in (3.4) at  $TG^{-1} \log T$  with a negligible error. Besides (5.2) we need one more ingredient from spectral theory, namely the author's bound [14]

$$(5.3) \quad \sum_{K-1 \leq \kappa_j \leq K+1} \alpha_j H_j^3(\tfrac{1}{2}) \ll_{\varepsilon} K^{1+\varepsilon}.$$

We give now the proof of Theorem when  $m = 2$  (the case  $m = 1$  easily follows from this and the Cauchy-Schwarz inequality). We use (4.7) with  $J_2$  replacing  $J_1$ . Then

$$\begin{aligned} \int_T^{2T} J_2^2(t, G) dt &\ll_{\varepsilon} T^{1+\varepsilon} + T^{\varepsilon} \max_{K \leq TG^{-1} \log T} \sum_{K < \kappa_j, \kappa_{\ell} \leq 2K} \alpha_j \alpha_{\ell} H_j^3(\tfrac{1}{2}) H_{\ell}^3(\tfrac{1}{2}) \times \\ &\times (\kappa_j \kappa_{\ell})^{-1/2} \left(\frac{\kappa_j}{4e}\right)^{i\kappa_j} \left(\frac{\kappa_{\ell}}{4e}\right)^{-i\kappa_{\ell}} \int_T^{2T} \varphi(t) t^{i\kappa_j - i\kappa_{\ell}} \exp\left(-\frac{1}{4} G^2 t^{-2} (\kappa_j^2 + \kappa_{\ell}^2)\right) dt. \end{aligned}$$

Repeated integrations by parts show that the contribution of  $\kappa_j, \kappa_{\ell}$  for which  $|\kappa_j - \kappa_{\ell}| \geq T^{\varepsilon}$  is negligible. The contribution of  $|\kappa_j - \kappa_{\ell}| < T^{\varepsilon}$  is estimated by (5.3) (splitting the summation over  $\kappa_{\ell}$  in subsums of length  $\leq 2$ ) and (5.2), while the integral over  $t$  is estimated trivially. The contribution will be

$$\begin{aligned} &\ll_{\varepsilon} T^{1+\varepsilon} \max_{K \leq TG^{-1} \log T} T^{-1} \sum_{K < \kappa_j \leq 2K} \alpha_j H_j^3(\tfrac{1}{2}) \kappa_j^{-1/2} \sum_{|\kappa_j - \kappa_{\ell}| < T^{\varepsilon}} \alpha_{\ell} H_{\ell}^3(\tfrac{1}{2}) \kappa_{\ell}^{-1/2} \\ &\ll_{\varepsilon} T^{\varepsilon} \max_{K \leq TG^{-1} \log T} \sum_{K < \kappa_j \leq 2K} \alpha_j H_j^3(\tfrac{1}{2}) K^{-1/2} K^{1/2} \\ &\ll_{\varepsilon} T^{2+\varepsilon} G^{-2} \ll_{\varepsilon} T^{1+\varepsilon} \end{aligned}$$

for  $G \geq T^{1/2+\varepsilon}$ , as asserted. We remark that the technique of this proof can be used, following the arguments in [9, Chapter 5], to yield a quick proof of the important bound

$$\int_0^T E_2^2(t) dt \ll_{\varepsilon} T^{2+\varepsilon},$$

which is only slightly weaker than the second bound in (3.3).

The cases  $m = 3$  and  $m = 4$  are dealt with analogously. For the former, after we raise the sum in (3.4) to the cube, it is seen that the non-negligible contribution comes from the triplets  $(\kappa_j, \kappa_m, \kappa_\ell)$  for which  $|\kappa_j + \kappa_m - \kappa_\ell| < T^\varepsilon$ . For the summation over one of the variables, say  $\kappa_\ell$ , we use (5.3), and for the summation over  $\kappa_j, \kappa_m$  we use (5.2). Similarly, in the case of the fourth power the non-negligible contribution will come from quadruples  $(\kappa_j, \kappa_m, \kappa_\ell, \kappa_n)$  for which

$$(5.4) \quad |\kappa_j + \kappa_m - \kappa_\ell - \kappa_n| < T^\varepsilon.$$

In this way the assertions of Theorem 2 concerning the cases  $m = 3, 4$  are obtained; the details are omitted for the sake of brevity. Note that  $\lambda_j = \kappa_j^2 + \frac{1}{4}$ , so that (5.4) can be rewritten as

$$(5.5) \quad |\sqrt{\lambda_j} + \sqrt{\lambda_m} - \sqrt{\lambda_\ell} - \sqrt{\lambda_n}| < T^\varepsilon,$$

which is somewhat analogous to (4.12). Lemma 5 provides a good bound for the number of integer quadruples satisfying (4.12), but the condition (5.5) is much more difficult to deal with, since little is known about arithmetic properties of the spectral values  $\lambda_j$ .

#### 4. BOUNDS FOR MOMENTS OF $|\zeta(\frac{1}{2} + it)|$

We shall show now how the results on power moments of  $|\zeta(\frac{1}{2} + it)|$  follow from mean square results on short intervals. In particular, a new result will be derived, which connects power moments of  $|\zeta(\frac{1}{2} + it)|$  with upper bounds furnished by Theorem 1 and Theorem 2.

To begin with, suppose that  $\{t_r\}_{r=1}^R$  are points lying in  $[T, 2T]$  such that  $t_{r+1} - t_r \geq 1$  ( $r = 1, \dots, R-1$ ) and  $|\zeta(\frac{1}{2} + it_r)| \geq V \geq T^\varepsilon$  for  $r = 1, \dots, R$ . From (1.5) we have

$$(6.1) \quad \begin{aligned} RV^{2k} &\ll \log T \sum_{r=1}^R \int_{t_r-1/3}^{t_r+1/3} |\zeta(\tfrac{1}{2} + it)|^{2k} dt \\ &\ll \log T \sum_{s=1}^S \int_{\tau_s-G}^{\tau_s+G} |\zeta(\tfrac{1}{2} + it)|^{2k} dt \\ &\ll G \log T \sum_{s=1}^S J_k(\tau_s, G), \end{aligned}$$

where we have grouped integrals over disjoint intervals  $[t_r - 1/3, t_r + 1/3]$  into integrals over disjoint intervals  $[\tau_s - G, \tau_s + G]$  with  $s = 1, \dots, S$  ( $S \leq R$ ),  $G \geq T^\varepsilon$ .

Suppose now that  $k = 1$  in (6.1). Then we use Lemma 2, noting that there are no absolute value signs on the right-hand side of (2.9), which can be truncated at  $TG^{-2} \log T$  and where, as before, we may assume that  $G \leq T^{1/3}$ . Exchanging the order of summation, it follows from (6.1) that

$$\begin{aligned}
 & RV^2 \ll RG \log^2 T + \\
 & + \sqrt{2}G \log T \sum_{n \leq TG^{-2} \log T} (-1)^n d(n) n^{-1/2} \times \\
 (6.2) \quad & \times \sum_{s=1}^S \left( \left( \frac{t_s}{2\pi n} + \frac{1}{4} \right) - \frac{1}{2} \right)^{-1/2} \exp(-G^2 \cdots) \sin f(t_s, n) \\
 & \ll RG \log^2 T + G \left( \sum_{n \leq TG^{-2} \log T} d^2(n) n^{-1} \right)^{1/2} \left( \sum_{n \leq TG^{-2} \log T} \left| \sum_{s=1}^S \cdots \right|^2 \right)^{1/2},
 \end{aligned}$$

by the Cauchy-Schwarz inequality. We further have

$$\sum_{n \leq TG^{-2} \log T} \left| \sum_{s=1}^S \cdots \right|^2 = \sum_{s_1, s_2 \leq S} S_0,$$

where

$$\begin{aligned}
 S_0 := & \sum_{n \leq TG^{-2} \log T} \left( \left( \frac{t_{s_1}}{2\pi n} + \frac{1}{4} \right)^{1/2} - \frac{1}{2} \right)^{-1/2} \\
 & \times \exp(-G^2 (\operatorname{arsinh} \sqrt{\pi n / (2t_{s_1})})^2) \left( \left( \frac{t_{s_2}}{2\pi n} + \frac{1}{4} \right)^{1/2} - \frac{1}{2} \right)^{-1/2} \\
 & \times \exp(-G^2 (\operatorname{arsinh} \sqrt{\pi n / (2t_{s_2})})^2) \exp(if(t_{s_1}, n) - if(t_{s_2}, n)).
 \end{aligned}$$

Removing by partial summation monotonic coefficients from  $S_0$ , we are led to the crucial exponential sum

$$(6.3) \quad S_1 := \sum_{n \leq M} \exp(if(t_{s_1}, n) - if(t_{s_2}, n)) \quad (M \leq TG^{-2} \log T).$$

The quality of the estimation of  $S_1$  is limited by the scope of the present-day exponential sum techniques (see e.g., M.N. Huxley [4]). The terms  $s_1 = s_2$  in (6.3) will eventually give rise to  $R \ll_{\varepsilon} T^{1+\varepsilon} V^{-6}$ , namely to a weak form of the sixth moment (3.6), but it does not seem likely that (3.6) can be reached (unconditionally) in this fashion. Observing that

$$(6.4) \quad \frac{\partial f(x, k)}{\partial x} = 2 \operatorname{arsinh} \sqrt{\frac{\pi k}{2x}} \sim \sqrt{\frac{2\pi k}{x}} \quad (x \asymp T, k \leq TG^{-2} \log T),$$

setting

$$f(u) := f(t_{s_1}, u) - f(t_{s_2}, u), \quad F := |t_{s_1} - t_{s_2}|(KT)^{-1/2},$$

we have (see [8, Chapters 1-2] for the relevant exponent pair technique) that  $S_1$  may be split into  $O(\log T)$  subsums of the type

$$\begin{aligned} \sum_{K < k \leq K' \leq 2K} \exp(if(k)) &\ll F^\kappa K^\lambda + F^{-1} \\ &\ll J^\kappa T^{-\kappa/2} K^{\lambda-\kappa/2} + (KT)^{1/2} |t_{s_1} - t_{s_2}|^{-1} \quad (K \leq TG^{-2} \log T), \end{aligned}$$

provided that  $|t_{s_1} - t_{s_2}| \leq J(\ll T)$ , and  $(\kappa, \lambda)$  is a (one-dimensional) exponent pair. Choosing  $(\kappa, \lambda) = (\frac{1}{2}, \frac{1}{2})$ ,  $J = T^{-\varepsilon} G^3$  we obtain (2.7) (with  $T^\varepsilon$  in place of  $\log^{17} T$ ). Namely with  $J = T^{-\varepsilon} G^3$  the number of points  $R = R_0$  to be estimated satisfies  $R_0 \ll_\varepsilon T^{1+\varepsilon} G^{-3}$ , hence dividing  $[T/2, 5T/2]$  into subintervals of length not exceeding  $J$  one obtains

$$R \ll R_0(1 + T/J) \ll_\varepsilon T^{2+\varepsilon} G^{-6} \ll_\varepsilon T^{2+\varepsilon} V^{-12},$$

which easily yields (2.7) (with  $T^\varepsilon$  in place of  $\log^{17} T$ ). This analysis was carried in detail in [8, Chapter 8], where the possibilities of choosing other exponent pairs besides  $(\kappa, \lambda) = (\frac{1}{2}, \frac{1}{2})$  were discussed.

Another type of a similar estimate was obtained by the author in [15] (for the analysis of sums of moments over well-spaced points  $\{t_r\} \in [T, 2T]$  the reader is referred to [10]). This result will be stated here as

**THEOREM 3.** *Let  $T \leq t_1 < \dots < t_R \leq 2T$  be points such that  $|\zeta(\frac{1}{2} + it_r)| \geq VT^{-\varepsilon}$  with  $t_{r+1} - t_r \geq V \geq T^{\frac{1}{10}+\varepsilon}$  for  $r = 1, \dots, R-1$ . Then, for any fixed integer  $M \geq 1$ ,*

$$\begin{aligned} (6.4) \quad R &\ll_\varepsilon T^{\varepsilon-M/2} V^{-2} \max_{K \leq T^{1+\varepsilon} V^{-4}} \times \\ &\times \int_{T/2}^{5T/2} \varphi(t) \left| \sum_{K \leq k \leq K' \leq 2K} (-1)^k d(k) k^{-1/4} \exp(2i\sqrt{2\pi kt} + cik^{3/2}t^{-1/2}) \right|^{2M} dt, \end{aligned}$$

where  $c = \sqrt{2\pi^3}/6$  and  $\varphi(t)$  is a non-negative, smooth function supported in  $[T/2, 5T/2]$  such that  $\varphi(t) = 1$  for  $T \leq t \leq 2T$ .

The case  $M = 1$  quickly leads to a weakened form of the fourth moment estimate, namely  $\int_0^T |\zeta(\frac{1}{2} + it)|^4 dt \ll_\varepsilon T^{1+\varepsilon}$ . The case  $M = 2$  of (6.4), by the use of Lemma 5 and Lemma 6, will lead again to a weakened form of the twelfth moment bound (2.7) (with  $T^{2+\varepsilon}$ ).



Finally we present a new result, which connects bounds for moments of  $|\zeta(\frac{1}{2} + it)|$  to bounds of moments of  $J_k(t, G)$ . This is

**THEOREM 4.** *Suppose that*

$$(6.5) \quad \int_T^{2T} J_k^m(t, G) dt \ll_\varepsilon T^{1+\varepsilon}$$

*holds for some fixed  $k, m \in \mathbb{N}$  and  $G \geq T^{\alpha_{k,m} + \varepsilon}$ ,  $0 \leq \alpha_{k,m} < 1$ . Then*

$$(6.6) \quad \int_0^T |\zeta(\frac{1}{2} + it)|^{2km} dt \ll_\varepsilon T^{1+(m-1)\alpha_{k,m} + \varepsilon}.$$

**Proof of Theorem 4.** We note first that ( $\mu(\cdot)$  denotes measure) the bound

$$(6.7) \quad \mu\left(t \in [T, 2T] : J_k(t, G) \geq U\right) \ll_\varepsilon T^{1+\varepsilon} U^{-m}$$

follows from (6.5). We use (6.1), dividing the sum over  $s$  into  $O(\log T)$  subsums where  $U < J_k(\tau_s, G) \leq 2U$ . Then, for  $U_0 (\gg 1)$  to be determined later,

$$\begin{aligned} \sum_{s=1}^S J_k(\tau_s, G) &\ll SU_0 + \log T \max_{U \geq U_0} \sum_{s, U < J_k(\tau_s, G) \leq 2U} J_k(\tau_s, G) \\ &\ll SU_0 + U \log T \max_{U \geq U_0} \sum_{s, U < J_k(\tau_s, G) \leq 2U} 1 \\ &\ll_\varepsilon SU_0 + G^{-1} \log T \max_{U \geq U_0} T^{1+\varepsilon} U^{1-m} \\ &\ll_\varepsilon SU_0 + T^{1+\varepsilon} U_0^{1-m} G^{-1}, \end{aligned}$$

since  $m \geq 1$  and

$$(6.8) \quad \sum_{s, J_k(\tau_s, G) > U} 1 \ll_\varepsilon T^{1+\varepsilon} U^{1-m} G^{-1}.$$

Namely if  $J_k(\tau_s, G) > U$ , then  $J_k(t, 2G) \geq U$  for  $t \in [\tau_s - \frac{1}{2}G, \tau_s + \frac{1}{2}G]$ , and (6.8) follows from (6.7). The choice

$$U_0 = \left(\frac{T}{SG}\right)^{1/m} (\gg 1)$$

yields

$$(6.9) \quad \sum_{s=1}^S J_k(\tau_s, G) \ll_\varepsilon T^{1/m+\varepsilon} S^{1-1/m} G^{-1/m}.$$

Inserting (6.9) in (6.1) we obtain

$$(6.10) \quad R \ll_{\varepsilon} T^{1+\varepsilon} G^{m-1} V^{-2km},$$

and (6.6) easily follows from (6.10), on taking  $G = T^{\alpha_{k,m}+\varepsilon}$ .

This completes the proof of Theorem 4. The values  $\alpha_{1,2} = 0$  (Theorem 1) and  $\alpha_{2,2} = \frac{1}{2}$  (Theorem 2) yield, respectively,

$$(6.11) \quad \int_0^T |\zeta(\tfrac{1}{2} + it)|^4 dt \ll_{\varepsilon} T^{1+\varepsilon}, \quad \int_0^T |\zeta(\tfrac{1}{2} + it)|^8 dt \ll_{\varepsilon} T^{3/2+\varepsilon}.$$

The bounds in (6.11) are, of course, well-known, but they are (up to the factor  $T^{\varepsilon}$ ) the sharpest known ones, and the bound for the fourth moment is essentially of the correct order of magnitude. Other values of  $\alpha_{k,m}$  ( $k = 1, 2$ ), furnished by Theorem 1 and Theorem 2, do not yield any new bounds, as can be readily checked. However, it seems that this approach is of interest, especially in view of recent results on the distribution of sums and differences of square roots of integers (cf. Lemma 4 and Lemma 5).

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